Classification of quasifinite representations of a Lie algebra related to Block type [1]

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Abstract. A well-known theorem of Mathieu's states that a Harish-chandra module over the Virasoro algebra is either a highest weight module, a lowest weight module or a module of the intermediate series. It is proved in this paper that an analogous result also holds for the Lie algebra $\mathcal{B}$ related to Block type, with basis $\{L_{\alpha,i}, C\,|\,\alpha, i\in\mathbb{Z}, i\geq 0\}$ and relations $[L_{\alpha,i},L_{\beta,j}] = ((i+1)\beta-(j+1)\alpha)L_{\alpha+\beta,i+j}+\delta_{\alpha+\beta,0}\delta_{i+j,0}\frac{\alpha^3-\alpha}{6}C$. Namely, an irreducible quasifinite $\mathcal{B}$-module is either a highest weight module, a lowest weight module or a module of the intermediate series.

Keywords: the Virasoro algebra; Block type Lie algebras; quasifinite representations

1. Introduction

Since a class of infinite dimensional simple Lie algebras was introduced by Block [1], generalizations of Lie algebras of this type (usually referred to as *Lie algebras of Block type*) have been studied by many authors (see for example, [2, 8, 12–20, 22–24]). Lie algebras of Block type are closely related to the Virasoro algebra, Virasoro-like algebra, or special cases of (generalized) Cartan type S Lie algebra or Cartan type H Lie algebra (e.g., [21]). It is well known that although Cartan type Lie algebras have a long history, their representation theory is however far from being well developed. In order to better understand the representation theory of Cartan type Lie algebras, it is very natural to first study representations of special cases of Cartan type Lie algebras. Partially due to these facts, the study of Lie algebras of this kind has recently attracted some authors' attentions.

The author in [12–14] presented a classification of the so-called *quasifinite modules* (which are simply $\mathbb{Z}$-graded modules with finite dimensional homogenous subspaces) over some Block type Lie algebras. In particular, he studied the representations of the Block type Lie algebra $\overline{\mathcal{B}}$ with basis $\{L_{\alpha,i}, C\,|\,\alpha, i\in\mathbb{Z}, i\geq -1\}$ over $\mathbb{C}$ and relations [12]

$$[L_{\alpha,i},L_{\beta,j}] = ((i+1)\beta-(j+1)\alpha)L_{\alpha+\beta,i+j}+\alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}C,\quad [C,L_{\alpha,i}]=0,$$

for $\alpha,\beta\in\mathbb{Z}, i,j\geq -1$. It is shown that quasifinite modules with dimensions of homogenous subspaces being uniformly bounded are all trivial — this renders the representations of this kind do not seem to be much interesting. On the other hand, the authors in [16] considered Verma type modules of the above Lie algebra $\overline{\mathcal{B}}$. However it turns out that these Verma

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type modules, regarded as $\mathbb{Z}$-graded modules, are all with infinite dimensional homogenous subspaces (except the subspace spanned by the generator which has dimension 1). Probably because of the above, the authors in [17] turned to study representations of the Lie algebra $\mathcal{B}$ related to Block type, which is a subalgebra of $\overline{\mathcal{B}}$, with basis $\{L_{\alpha,i}, C \,|\, \alpha, i \in \mathbb{Z}, i \geq 0\}$ and relations

$$[L_{\alpha,i}, L_{\beta,j}] = ((i+1)\beta - (j+1)\alpha)L_{\alpha+\beta,i+j} + \delta_{\alpha+\beta,0}\delta_{i+j,0}\frac{\alpha^3-\alpha}{6}C. \tag{1.1}$$

This Lie algebra $\mathcal{B}$, as stated in [17], is interesting in the sense that it contains the following subalgebra

$$\mathrm{Vir} = \mathrm{span}\{L_{\alpha,0}, C \,|\, \alpha \in \mathbb{Z}\}, \tag{1.2}$$

which is isomorphic to the well-known Virasoro algebra (cf. (2.1)), while the Lie algebra $\overline{\mathcal{B}}$ contains no such a subalgebra. Due to this, one will see that the representation theory for the Lie algebra $\mathcal{B}$ is quite different from that for $\overline{\mathcal{B}}$. Moreover, $\mathcal{B}$ is related to the well-known W-infinity Lie algebra $\mathcal{W}_\infty$ (e.g., [15]) in the following way: Recall that the W-infinity Lie algebra $\mathcal{W}_{1+\infty}$ is defined to be the universal central extension of infinite dimensional Lie algebra of differential operators on the circle, which has a basis $\{x^\alpha D^i, C| \, \alpha, i \in \mathbb{Z}, i \geq 0\}$ with $D = \frac{d}{dx}$, and relations

$$[x^\alpha D^i, x^\beta D^j] = x^{\alpha+\beta}((D+\beta)^i D^j - D^i(D+\alpha)^j) + \delta_{\alpha+\beta,0}(-1)^i i! j! \binom{\alpha+i}{i+j+1} C.$$

Then the W-infinity algebra $\mathcal{W}_\infty$, the universal central extension of infinite dimensional Lie algebra of differential operators on the circle of degree at least one, is simply the subalgebra of $\mathcal{W}_{1+\infty}$ spanned by $\{x^\alpha D^i, C \,|\, \alpha, i \in \mathbb{Z}, i \geq 1\}$. If we define a natural filtration of $\mathcal{W}_\infty$ by

$$\{0\} = (\mathcal{W}_\infty)_{[-2]} \subset (\mathcal{W}_\infty)_{[-1]} \subset \cdots \subset \mathcal{W}_\infty,$$

where $(\mathcal{W}_\infty)_{[-1]} = \mathbb{C}C$, and $(\mathcal{W}_\infty)_{[n]} = \mathrm{span}\{x^\alpha D^i, C \,|\, \alpha \in \mathbb{Z},\ i \leq n+1\}$ for $n \geq 0$, then $\mathcal{B}$ is simply the associated graded Lie algebra of the filtered Lie algebra $\mathcal{W}_\infty$. As stated in [9, 11, 15], the W-infinity algebras arise naturally in various physical theories, such as conformal field theory, the theory of the quantum Hall effect, etc.; among them the $\mathcal{W}_\infty$ algebra and $\mathcal{W}_{1+\infty}$ algebra, of interest to both mathematicians and physicists, have received intensive studies in the literature.

We find the Lie algebra $\mathcal{B}$ is interesting in another aspect that it also contains many (finitely) $\mathbb{Z}$-graded subquotient algebras $\tilde{\mathcal{B}}_{m,n}$ for $n \geq m \geq 0$, where

$$\tilde{\mathcal{B}}_{m,n} = \mathcal{B}_m/\mathcal{B}_{n+1}, \quad \mathcal{B}_m = \mathrm{span}\{L_{a,i} \,|\, \alpha \in \mathbb{Z},\ i \geq m\}. \tag{1.3}$$

For instance, $\tilde{\mathcal{B}}_{0,0}$ is the Virasoro algebra (thus the Virasoro algebra is both a subalgebra and a quotient algebra of $\mathcal{B}$), $\tilde{\mathcal{B}}_{0,1}$ is the W-algebra $W(-2,0)$, and $\tilde{\mathcal{B}}_{1,2}$ is the Lie algebra

of invariance of the free Schröinger equation (and thus $\tilde{\mathcal{B}}_{0,2}$ is closely related to the twisted Schröinger-Virasoro algebra). It is well-known that the category of quasifinite modules (cf. Definition 2.1) over a quotient Lie algebra $\tilde{\mathcal{B}}_{0,n}$ is a full subcategory of the category of quasifinite $\mathcal{B}$-modules. Thus a classification of irreducible quasifinite $\mathcal{B}$-modules also gives a classification of irreducible quasifinite $\tilde{\mathcal{B}}_{0,n}$-modules for all $n > 0$.

The authors in [17] prove that an irreducible quasifinite $\mathcal{B}$-module is either a highest weight module, a lowest weight module or a uniformly bounded module. They also obtain a complete description of irreducible quasifinite highest weight modules and modules of the intermediate series over $\mathcal{B}$. Motivated by a well-known result of Mathieu's in [5], it is very natural to consider the classification of irreducible quasifinite $\mathcal{B}$-modules. In this paper, by proving that an irreducible quasifinite uniformly bounded $\mathcal{B}$-module is a module of the intermediate series, we obtain the following main result.

Theorem 1.1 *Any irreducible quasifinite $\mathcal{B}$-module is either a highest weight module, a lowest weight module or a module of the intermediate series.*

As a by-product of our proof, we also obtain the following, which recovers a result in [17, Theorem 4.8].

Corollary 1.2 *A module of the intermediate series over $\mathcal{B}$ is simply a module of the intermediate series over the Virasoro algebra Vir with the trivial action of $\mathcal{B}_1$.*

The proof of [17, Theorem 4.8] given in [17] involves some heavy computations. In our case here, some new techniques are employed, which renders the proof seems to be more elegant, as one may see in Section 3. We would like to point out that the techniques used here may be used to dealing analogous problems of Lie algebras which are closely related to the Lie algebra $\mathcal{B}$. This is also our main motivation to present this paper.

Since lowest weight modules are duals of highest weight modules, Theorem 1.1 together with Corollary 1.2 gives a complete classification of irreducible quasifinite $\mathcal{B}$-module. The analogous results to the this theorem for the Virasoro algebra, W-infinity algebras, higher rank Virasoro algebras, and some Lie algebras of Block type were obtained in [4, 5, 10–15].

2. Preliminaries

It is well-known that the Virasoro algebra Vir is the Lie algebra with basis $\{L_i, C \mid i \in \mathbb{Z}\}$ satisfying the relations

$$[L_i, L_j] = (j-i)L_{i+j} + \frac{i^3 - i}{12}\delta_{i,-j}C, \qquad [L_i, C] = 0 \quad \text{for} \quad i, j \in \mathbb{Z}. \tag{2.1}$$

A Vir-module of the intermediate series must be one of $A_{a,b}, A(a), B(a), a, b \in \mathbb{C}$, or one of the quotient submodules, where $A_{a,b}, A(a), B(a)$ all have a basis $\{x_k \mid k \in \mathbb{Z}\}$ such that C

acts trivially and

$$
\begin{aligned}
&A_{a,b}: \ L_i x_k = (a+k+bi)x_{i+k}, && && (2.2)\\
&A(a): L_i x_k = (i+k)x_{i+k} \quad (k \neq 0), && L_i x_0 = i(i+a)x_i, && (2.3)\\
&B(a): L_i x_k = k x_{i+k} \quad (k \neq -i), && L_i x_{-i} = -i(i+a)x_0, && (2.4)
\end{aligned}
$$

for $i, k \in \mathbb{Z}$. One also has

$$
\begin{aligned}
&A_{a,b} \text{ is irreducible} \iff a \notin \mathbb{Z} \text{ or } a \in \mathbb{Z},\ b \neq 0,1,\\
&A_{a,1} \cong A_{a,0} \text{ if } a \notin \mathbb{Z},\\
&A_{a,b} \text{ is reducible, and } A'_{a,1} \cong A'_{a,0} \cong A'_{0,0} \text{ if } a \in \mathbb{Z},\ b = 0,1,
\end{aligned}
$$

where in general, $A'_{a,b}$ denotes the unique nontrivial composition factor of $A_{a,b}$.

Now consider the Lie algebra $\mathcal{B}$. We can realize it in the space $\mathbb{C}[x, x^{-1}] \otimes t\mathbb{C}[t] \oplus \mathbb{C}C$ with the bracket

$$
[x^\alpha f(t), x^\beta g(t)] = x^{\alpha+\beta}(\beta f'(t)g(t) - \alpha f(t)g'(t)) + \delta_{\alpha+\beta,0}\frac{\alpha^3-\alpha}{6}\mathrm{Res}_t t^{-3} f(t)g(t)C, \quad (2.5)
$$

for $\alpha, \beta \in \mathbb{Z}$, $f(t), g(t) \in t\mathbb{C}[t]$, where the prime stands for $\frac{d}{dt}$, and $\mathrm{Res}_t f(t)$ stands for the residue of the Laurent polynomial $f(t)$, namely the coefficient of t^{-1} in $f(t)$. The Lie algebra $\mathcal{B}$ has a natural $\mathbb{Z}$-gradation $\mathcal{B} = \oplus_{\alpha\in\mathbb{Z}}\mathcal{B}_\alpha$ with $\mathcal{B}_\alpha = \{x^\alpha f(t) \,|\, f(t) \in t\mathbb{C}[t]\} + \delta_{\alpha,0}\mathbb{C}C$. Putting $\mathcal{B}_\pm = \oplus_{\pm\alpha>0}\mathcal{B}_\alpha$, we have the triangular decomposition $\mathcal{B} = \mathcal{B}_- \oplus \mathcal{B}_0 \oplus \mathcal{B}_+$. Note that $\mathcal{B}_0 = t\mathbb{C}[t] \oplus \mathbb{C}C$ is an infinite dimensional commutative subalgebra of $\mathcal{B}$ (but not a Cartan subalgebra). When we study representations of a Lie algebra of this kind, as pointed in [12–14], we encounter the difficulty that though it is $\mathbb{Z}$-graded, the graded subspaces are still infinite dimensional, thus the study of quasifinite modules is a nontrivial problem.

Denote

$$
L_\alpha = x^\alpha, \quad L_{\alpha,i} = x^\alpha t^{i+1} \quad \text{for} \quad \alpha, i \in \mathbb{Z},\ i \geq 0. \quad (2.6)
$$

Then (2.5) is equivalent to (1.1) and $\mathrm{span}\{L_\alpha, C \,|\, \alpha \in \mathbb{Z}\} \cong \mathrm{Vir}$ (cf. (1.2) and (2.1)).

Definition 2.1 A module V over $\mathcal{B}$ is called

- *$\mathbb{Z}$-graded* if $V = \oplus_{\alpha\in\mathbb{Z}}V_\alpha$ and $\mathcal{B}_\alpha V_\beta \subset V_{\alpha+\beta}$ for all α, β;
- *quasifinite* if it is $\mathbb{Z}$-graded and $\dim V_\beta < \infty$ for all β;
- *uniformly bounded* if it is $\mathbb{Z}$-graded and there is $N > 0$ such that $\dim V_\beta \leq N$ for all β;
- a *module of the intermediate series* if it is $\mathbb{Z}$-graded and $\dim V_\beta \leq 1$ for all β;
- a *highest* (respectively *lowest*) *weight module* if there exists some $\Lambda \in \mathcal{B}_0^*$ (the dual space of $\mathcal{B}_0$) such that $V = V(\Lambda)$, where $V(\Lambda)$ is a module generated by a *highest* (respectively *lowest*) *weight vector* $v_\Lambda \in V(\Lambda)_0$, i.e., v_Λ satisfies

$$
hv_\Lambda = \Lambda(h)v_\Lambda \ \text{ where } \ h \in \mathcal{B}_0, \quad \text{and} \quad \mathcal{B}_+ v_\Lambda = 0 \ \text{ (respectively } \mathcal{B}_- v_\Lambda = 0).
$$

Suppose now $V = \oplus_{k\in\mathbb{Z}} V_k$ is an irreducible uniformly bounded $\mathcal{B}$-module. For $a \in \mathbb{C}$, we let $V[a] = \oplus_{k\in\mathbb{Z}} V_k[a]$, where $V_k[a] = \{v \in V_k \mid L_0 v = (a+k)v\}$. Then obviously, $V[a]$ as a $\mathcal{B}$-submodule is a direct summand of V. Thus $V = V[a]$ for some fixed $a \in \mathbb{C}$, that is to say,

$$V = \bigoplus_{k\in\mathbb{Z}} V_k, \ \text{ where } V_k = \{v \in V \mid L_0 v = (a+k)v\} \ (\textit{weight space with weight } a+k). \tag{2.7}$$

Note that regarding as a Vir-module, V is also a uniformly bounded Vir-module. Therefore by the result of [6, 7], we have the following lemma.

Lemma 2.2 *If V is an irreducible uniformly bounded $\mathcal{B}$-module as in* (2.7)*, then there exists a non-negative integer N such that* $\dim V_k[a] = N$ *for all $k \in \mathbb{Z}$ with $k + a \neq 0$.*

3. Proof of the main theorem

The aim of this section is to prove Theorem 1.1. Let a be fixed such that (2.7) holds (we always choose a to be zero if $a \in \mathbb{Z}$). For any $k \in \mathbb{Z}$, the following notation will be frequently used,

$$\tilde{k} = a + k. \tag{3.1}$$

Let V as in (2.7) be an irreducible uniformly bounded $\mathcal{B}$-module. By Lemma 2.2, $\dim V_k = N$ for all $k \in \mathbb{Z}$ with $\tilde{k} \neq 0$. Without loss of generality, we can suppose $N \geq 1$. Regarding V as a Vir-module and choosing a composition series, by a well-known theorem of Matheiu's [5], we can take a basis $Y_k = (y_k^{(1)}, ..., y_k^{(N)})$ of V_k with $\tilde{k} \neq 0$ satisfying

$$L_\alpha Y_k = Y_{\alpha+k} A_{\alpha,k} \ \text{ for } \ \alpha, k \in \mathbb{Z} \ \text{ with } \ \tilde{k},\, \alpha + \tilde{k} \neq 0, \tag{3.2}$$

such that $A_{\alpha,k}$ is an upper-triangular matrix with diagonals being $\tilde{k} + b_p\alpha$ for $p = 1, ..., N$ and some $b_p \in \mathbb{C}$. Our first result is the following.

Lemma 3.1 (1) *For all $i \gg 0$, the action of $L_{1,i}$ on V is trivial, i.e., $L_{1,i}|_V = 0$.*
(2) *There exists some $j \geq 0$ such that $\mathcal{B}_{j+1} V = 0$, where $\mathcal{B}_{j+1}$ is defined in* (1.3).

Proof. (1) Fix $i > 0$ and suppose $L_{1,i} Y_k = Y_{k+1} T_k$ for $\tilde{k}, \tilde{k}+1 \neq 0$ and some $N \times N$ matrix $T_k = (t_k^{p,q})_{p,q=1}^N$ (the symbol T_k remains to be undefined if $\tilde{k} = 0, -1$). Assume $T_k \neq 0$ and let (p, q) be (first) the leftmost and (then) the lowermost position such that

$$t_k \neq 0 \ \text{ for some } \ \tilde{k}, \tilde{k}+1 \neq 0, \tag{3.3}$$

where in general, we denote $t_j = t_j^{p,q}$ for all possible j. Let $\alpha, \beta \gg 0$ be such that $T_k, T_{\alpha+k}, T_{\beta+k}, T_{\alpha+\beta+k}$ appearing below are all defined. Applying the equation

$$\Big(1-(i+1)(\alpha+\beta)\Big)[L_\alpha,[L_\beta,L_{1,i}]] = \Big(1-(i+1)\beta\Big)\Big(1+\beta-(i+1)\alpha\Big)[L_{\alpha+\beta},L_{1,i}], \tag{3.4}$$

to Y_k, we obtain

$$\Big(1-(i+1)(\alpha+\beta)\Big)\Big(A_{\alpha,1+\beta+k}(A_{\beta,1+k}T_k-T_{\beta+k}A_{\beta,k})-(A_{\beta,1+\alpha+k}T_{\alpha+k}-T_{\alpha+\beta+k}A_{\beta,\alpha+k})A_{\alpha,k}\Big)$$
$$=\Big(1-(i+1)\beta\Big)\Big(1+\beta-(i+1)\alpha\Big)(A_{\alpha+\beta,1+k}T_k-T_{\alpha+\beta+k}A_{\alpha+\beta,k}). \tag{3.5}$$

Comparing the (p,q)-entry, we have

$$\Big(1-(i+1)(\alpha+\beta)\Big)\Big((1+\beta+\tilde{k}+b_q\alpha)\big((1+\tilde{k}+b_q\beta)t_k-t_{\beta+k}(\tilde{k}+b_p\beta)\big)$$
$$-\big((1+\alpha+\tilde{k}+b_q\beta)t_{\alpha+k}-t_{\alpha+\beta+k}(\alpha+\tilde{k}+b_p\beta)\big)(\tilde{k}+b_p\alpha)\Big)$$
$$=\Big(1-(i+1)\beta\Big)\Big(1+\beta-(i+1)\alpha\Big)\Big(\big(1+\tilde{k}+b_q(\alpha+\beta)\big)t_k-t_{\alpha+\beta+k}\big(\tilde{k}+b_p(\alpha+\beta)\big)\Big). \tag{3.6}$$

Setting the triple (α,β,k) in (3.6) to be $(\alpha,\alpha,k-\alpha)$, $(\alpha,-\alpha,k)$, $(-\alpha,-\alpha,k+\alpha)$ respectively, we obtain a system of three equations on variable $t_{k-\alpha},t_k,t_{k+\alpha}$. Note that the determinant $\Delta_{i,\alpha,k}$ of coefficients of this system is a polynomial on i,α,k with the degree of i being ≤ 6. If we compute the coefficient of i^6 in $\Delta_{i,\alpha,k}$, then only those terms in (3.6) with factor i^2 need to be considered, in this case, (3.6) can be simplified as

$$0=i^2\beta\alpha\Big(\big(1+\tilde{k}+b_q(\alpha+\beta)\big)t_k-t_{\alpha+\beta+k}\big(\tilde{k}+b_p(\alpha+\beta)\big)\Big).$$

From this, one can easily compute that the coefficient of i^6 in $\Delta_{i,\alpha,k}$ is $1+2(\alpha+\tilde{k})-4\alpha^2(b_p+b_p^2+b_q-b_q^2)$, which is nonzero when $\alpha\gg 0$. Therefore, $\Delta_{i,\alpha,k}\neq 0$ when $i,\alpha\gg 0$, and we obtain $t_k=0$ if $i\gg 0$, a contradiction with (3.3). This proves $L_{1,i}V_k=0$ for $\tilde{k},\tilde{k}+1\neq 0$ and $i\gg 0$, and in particular, we have (1) if $a\notin\mathbb{Z}$.

Now assume $a=0$. Take a basis $Y_0=(y_0^{(1)},...,y_0^{(N')})$ of V_0 (assume $\dim V_0=N'$), and assume $L_{1,i}Y_j=Y_0T_j$, $j=-1,0$, for some $N'\times N$ matrix T_{-1} and some $N\times N'$ matrix T_0. Applying (3.4) to $Y_{-\alpha-\beta-1}$ and Y_0, we obtain respectively (cf. (3.5))

$$T_{-1}P_{\alpha,\beta,i}=0,\quad Q_{\alpha,\beta,i}T_0=0, \tag{3.7}$$

where

$$P_{\alpha,\beta,i}=\Big(1-(i+1)(\alpha+\beta)\Big)A_{\beta,-\beta-1}A_{\alpha,-\alpha-\beta-1}+\Big(1-(i+1)\beta\Big)\Big(1+\beta-(i+1)\alpha\Big)A_{\alpha+\beta,-\alpha-\beta-1},$$
$$Q_{\alpha,\beta,i}=\Big(1-(i+1)(\alpha+\beta)\Big)A_{\alpha,1+\beta}A_{\beta,1}-\Big(1-(i+1)\beta\Big)\Big(1+\beta-(i+1)\alpha\Big)A_{\alpha+\beta,1},$$

which are upper-triangular matrices with nonzero diagonals when $\alpha,\beta,i\gg 0$. Thus $T_0=T_{-1}=0$ by (3.7).

(2) By (1), there exists j such that $L_{1,j+1}|_V=0$. Since $\mathcal{B}_{j+1}$ is an ideal of $\mathcal{B}$ generated (as an ideal) by $L_{1,j+1}$, we obtain (2). $\square$

Lemma 3.2 *We have $\mathcal{B}_1 V = 0$.*

Proof. Let j be the smallest such that Lemma 3.1(2) holds. Assume $j > 0$. Then there exists some $\alpha \in \mathbb{Z}$ with

$$L_{\alpha,j}|_V \neq 0, \tag{3.8}$$

and V becomes an irreducible module over $\tilde{\mathcal{B}}_{0,j}$ (cf. (1.3)). We shall use $\tilde{L}_{\alpha,i}$, $\alpha \in \mathbb{Z}$, $0 \le i \le j$, to denote basis elements of $\tilde{\mathcal{B}}_{0,j}$ (and still use L_α to denote $\tilde{L}_{\alpha,0}$). Denote $\tilde{\mathcal{C}} = \mathrm{span}\{\tilde{L}_{\alpha,j} \,|\, \alpha \in \mathbb{Z}\}$, which is an ideal of $\tilde{\mathcal{B}}_{0,j}$ and is in the center of $\tilde{\mathcal{B}}_{1,j}$. Then (3.8) shows $\tilde{\mathcal{C}}V$ is a nonzero submodule of V. Thus

$$\tilde{\mathcal{C}}V = V. \tag{3.9}$$

Take the subspace $M = \oplus_{i=-2}^{2} V_i$ of V. Then $\tilde{L}_{0,j}|_M$ is a linear transformation on the finite-dimensional space M. Let $f(\lambda)$ be its characteristic polynomial (or its minimal polynomial), so

$$f(\tilde{L}_{0,j})M = 0. \tag{3.10}$$

From (2.2)–(2.4), one can easily obtain by induction on N that

$$V = M + \mathrm{Vir}\, M, \tag{3.11}$$

which can also be obtained by proving by induction on q that every basis element $y_k^{(q)} \in V_k$ for all k with $|k| > 2$ is a linear combination of $L_{k+i} y_{-i}^{(p)}$ for $-2 \le i \le 2$ and $1 \le p \le q$. Noting that $\tilde{L}_{0,j}$ is in the center of the universal enveloping algebra $U(\tilde{\mathcal{B}}_{1,j})$ of $\tilde{\mathcal{B}}_{1,j}$, for any polynomial $g(\lambda)$ and $\alpha \in \mathbb{Z}$, we have (cf. (1.1))

$$g(\tilde{L}_{0,j})L_\alpha = L_\alpha\, g(\tilde{L}_{0,j}) + [g(\tilde{L}_{0,j}), L_\alpha] = L_\alpha\, g(\tilde{L}_{0,j}) + (j+1)\alpha g^{(1)}(\tilde{L}_{0,j})\tilde{L}_{\alpha,j}, \tag{3.12}$$

where in general, $g^{(p)}(\lambda)$ denotes the p-th derivative of $g(\lambda)$. Now take $g(\lambda) = f(\lambda)^2$. By (3.10) and (3.12), we have

$$g(\tilde{L}_{0,j})L_\alpha M \subset 2(j+1)\alpha \tilde{L}_{\alpha,j} f^{(1)}(\tilde{L}_{0,j}) f(\tilde{L}_{0,j})M = 0 \quad \text{for} \quad \alpha \in \mathbb{Z}, \tag{3.13}$$

which together with (3.10) and (3.11) proves $g(\tilde{L}_{0,j})V = 0$. From this, as in (3.12), by considering $g(\tilde{L}_{0,j})L_{\alpha_1} \cdots L_{\alpha_q}$ and induction on q, we have

$$g^{(q)}(\tilde{L}_{0,j})\tilde{L}_{\alpha_1,j} \cdots \tilde{L}_{\alpha_q,j} V = 0, \tag{3.14}$$

for all $\alpha_1, ..., \alpha_q \in \mathbb{Z}\backslash\{0\}$. If, say, $\alpha_1 = 0$, by choosing some $\alpha \in \mathbb{Z}\backslash\{-\alpha_2, ..., -\alpha_q\}$, and using $L_{0,j} = ((j+1)\alpha)^{-1}[L_\alpha, L_{-\alpha,j}]$, we see that (3.14) still holds. From this, we see that (3.14) holds for all $\alpha_i \in \mathbb{Z}$. This in particular proves that for $s = \deg g(\lambda)$,

$$\tilde{\mathcal{C}}^s V = 0. \tag{3.15}$$

This together with (3.9) is a contradiction. Thus $j = 0$ and the result is proved. $\square$

Lemma 3.2 says that V is simply an irreducible module over Vir, thus it is a module of the intermediate series. This completes the proof of Theorem 1.1.

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